**Induced Phase Shifts in Arbitrarily Bent Rectangular-Core Dual-Mode Waveguides**

Arun Kumar, Rajeev Jindal, and Robert L. Gallawa

**Abstract**—We examine the variation of the effective indexes of the two modes of an arbitrarily bent dual-mode rectangular-core waveguide. We find that under the large bending radius approximation, which is indeed the practical case for most of the devices, the waveguide bent with bending radius \( p \) in a plane at an angle \( \theta \) with the major axis is almost equivalent to bending it simultaneously in the plane of major and minor axes with bending radii \( p_{sec} \theta \) and \( p_{ose} \), respectively. The bending-induced phase difference between the two modes is found to be maximum when the waveguide is bent along the major axis, and to decrease first and then increase in the opposite direction as the \( V \)-number is decreased. The results of our study can be used to improve the sensitivity of the dual-mode optical waveguide sensors and devices based on the bending of fiber.

**I. INTRODUCTION**

Dual-mode elliptical-core fibers are increasingly being used to realize novel fiber-optic components and sensors [11]-[5]. Bending-induced changes such as mode-losses and phase-shifts are important design parameters for many such devices. Since the exact analysis available for elliptical core fiber is very complicated [6], it is often modeled by appropriate rectangular core waveguides (RCW’s) [7], [8]. Thus, a knowledge of bending-induced changes in the modal characteristics of an RCW is extremely important. Earlier analyses [9]-[11] reporting on bent RCW’s are restricted only to the bending in a plane containing major or the minor axis, while in practice the fiber may be bent in an arbitrary plane. For example, in the accelerator reported by Castro et al. [8], an elliptical core fiber is bent simultaneously about both the axes resulting in a continuous rotation of the bending plane as one moves along the fiber. In the present paper, we report a perturbation method to analyze an RCW bent in an arbitrary plane. The only assumption taken here is that the bending radius \( p \) is much larger than the core dimensions, which is the practical case most of the times.

**11. ANALYSIS**

We consider an RCW with major axis \( 2a \) and minor axis \( 2b \) bent along a circular arc of radius \( p \) in a plane containing \( 2' \)-axis which is at an angle \( \theta \) with the major axis (2) of the RCW (Fig. 1). The scalar modes are obtained by solving the following equation:

\[
\left[ \frac{\partial^2}{\partial y'^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial y'^2} + k_0^2 n_2 (r, y') - \frac{\beta^2 p^2}{r^2} \right] \Psi(r, y') = 0
\]

where the total field is assumed to be of the form

\[
E(r, y') = *n(r, y') \exp[i\beta p y']
\]

where \( n'(r, y') \) represents the relative dielectric constant profile of the waveguide, \( p \) represents the propagation constant of its modes, \( k_0 \) represents the free space wave number, and \( r, y' \) are the cylindrical coordinates as shown in Fig. 1.

Using the transformations

\[
\begin{align*}
(x', y') &= r^{1/2} U(r, y') \\
2' &= T - p
\end{align*}
\]

transforms to

\[
\frac{\partial^2}{\partial z'^2} + \frac{\partial^2}{\partial y'^2} + k_0^2 n(r, y') - \frac{\beta^2 p^2}{r^2} U'' = 0.
\]

In the above equations, we have assumed that \( p >> a \) and neglected \( 1/p^2 \) in comparison to \( n'(= \beta/k_0) \) which is indeed valid for large bending radii.

We further use the transformation relationship between \((z', v')\), and \((z, v)\), namely,

\[
\begin{align*}
x &= x' \cos 9 - y' \sin 9 \\
y &= x' \sin 9 + y' \cos 9
\end{align*}
\]
which transforms (3) into
\[
\frac{d^2 U}{dx^2} + \frac{d^2 U}{dy^2} + k_0^2 n^2 \int_{Q}^{Q+\sin 0} \frac{2}{dU} = 0. \tag{4}
\]

Here, \(n_2(x, y)\) represents the dielectric constant distribution of the straight RCW and \(U(x, y) = U'(x', y')\). Following earlier analyses \[11, 12\], we approximate \(n_2(x, y)\) by a separable profile,
\[
n_i(x, y) = n'_i(x) + n''_i(y) - n_2\tag{5}
\]
where \(n'_i(x) = n_1\) for \(1 \leq x \leq a\) and \(n_2\) for \(x > a\) and \(n''_i(y) = n_1\) for \(1 \leq y \leq b\) and \(n_2\) for \(y > b\). It should be noted that dielectric constant profile \(n_i(x, y)\) differs from the original profile \(n_2(x, y)\) only in the corner regions, and since the modal field power in the corner regions is small, one can consider this difference as a perturbation to this profile. For the unperturbed profile \(n_i(x, y)\), (4) can be solved exactly taking \(U(x, y)\) of the form
\[
U(X, Y) = X(X)Y(Y). \tag{6}
\]

It can be shown easily that \(X(x)\) and \(Y(y)\) are the solutions of the following one-dimensional equations:
\[
\frac{1}{X} \frac{d}{dx} \left( X \frac{dX}{dx} + k_0^2 n_i^2(x) \right) - \frac{P_i}{w_x} \left( \frac{2 \tan 8}{\sec^2 8} \right) X = 0 \tag{7}
\]

\[
\frac{1}{Y} \frac{d}{dy} \left( Y \frac{dY}{dy} + k_0^2 n_i^2(y) \right) - \frac{P_y}{w_y} \left( \frac{2 \tan 8}{\sec^2 8} \right) Y = 0 \tag{8}
\]
where \(\beta_i\) and \(P_i\) are two constants such that
\[
n_i^2 = \beta_i^2 k_0^2 = (\beta_x^2 + \beta_y^2 - k_0^2 n_i^2)/k_0^2. \tag{9}
\]

Using (9) in (7) and (8) they take the form
\[
\frac{d^2 X}{dx^2} + k_0^2 n_i^2 x - P_i \left( \frac{2 \tan 8}{\sec^2 8} \right) X = 0 \tag{10}
\]

\[
\frac{6 \beta X}{dy^2} + k_0^2 \left[ n_i^2 (y) - \beta_i^2 \left( \frac{2 \tan 8}{\sec^2 8} \right) \right] Y = 0. \tag{11}
\]

The values of \(B_i\) and \(P_i\) are obtained by solving (10) and (11), respectively, which are similar to the equations of bent planar waveguides described by \(n'(x)\) and \(n''(y)\) with bending radii \(p\) sec \(8\) and \(p\) cosec \(0\), respectively. The only difference is the last term in both the equations which is very small as compared to the earlier two terms because \(P_i/k_0 \cdot M_i\) is small and \(P >> a\). Thus, an arbitrarily bent RCW is almost equal to bending it simultaneously along the major and the minor axis with bending radii \(p\) sec \(0\) and \(p\) cosec \(0\), respectively.

For solving (10) and (11), we have used the perturbation approach similar to the one used by Takuma \(13\) for bent planar waveguide which, when tested with respect to the more accurate Airy function approach given by Goyal \(14\) for a bent planar waveguide, was found to give correct values of effective indexes up to 0.001%. According to the perturbation approach \(13\), the fields \(X(x), Y(y)\) and propagation constants \(P_x, P_y\) are expanded as a function of bending radii as
\[
\beta_x = \beta_{0x} + \frac{1}{P} \beta_x^2 \beta_{2x} - \beta_{2x} + \cdots \tag{13}
\]

\[
\beta_y = \beta_{0y} + \frac{1}{P} \beta_y^2 \beta_{2y} + \cdots \tag{14}
\]

Putting these equations back in (10) and (11), we obtain the first and the second order corrections in \(P_x, P_y\) (see Appendix) as
\[
\delta_{P_x} = 0, \delta_{P_y} = 0 \tag{15}
\]

where
\[
\begin{align*}
V_x & = \frac{V_0}{1 + \frac{w_x}{w_x}} \left( 1 \pm \frac{\beta_x}{w_x} \right) \\
V_y & = \frac{V_0}{1 + \frac{w_y}{w_y}} \left( 1 \pm \frac{\beta_y}{w_y} \right)
\end{align*} \tag{16}
\]

The functions \(F_x\) and \(F_y\) are given by
\[
F_x = \frac{B_i V_x}{w_x} \left[ 1 - \frac{5 V_x}{w_x} + \frac{3}{2 w_x} \left( 1 - \frac{5}{4 w_x} \right) + \frac{V_x}{w_x} \left( 1 - \frac{3}{4 w_x} \right) \right]
\]

\[
F_y = \frac{B_i V_y}{w_y} \left[ 1 - \frac{5 V_y}{w_y} + \frac{3}{2 w_y} \left( 1 - \frac{5}{4 w_y} \right) + \frac{V_y}{w_y} \left( 1 - \frac{3}{4 w_y} \right) \right]
\]
\[
X' = \left[ \begin{array}{c}
\frac{\beta_0^2}{2v_x^2} \cos \theta + \frac{\beta_0^2}{2v_y^2} \cos \phi_x \left( \frac{1}{1 + \frac{v_x^2}{w_x^2}} - \left( \frac{\beta_0^2}{2v_x^2} \right)^2 \sin \left( \frac{\beta_0^2}{2v_x^2} \phi_x \right) \right) \exp \left[ -\frac{w_x^2}{a}(x-a) \right] \\
\frac{\beta_0^2}{2v_y^2} \cos \theta + \frac{\beta_0^2}{2v_y^2} \cos \phi_y \left( \frac{1}{1 + \frac{v_y^2}{w_y^2}} - \left( \frac{\beta_0^2}{2v_y^2} \right)^2 \sin \left( \frac{\beta_0^2}{2v_y^2} \phi_y \right) \right) \exp \left[ -\frac{w_y^2}{a}(y-a) \right]
\end{array} \right]
\]

with

\[
\Delta n_0^2 = \frac{\int \int \left[ n_0^2 - \frac{n_2^2}{2} \right]|U(x,y)|^2 dx dy}{A_w}
\]

where \( A_w \) and \( A_c \) are the area of the corner regions and the entire waveguide, respectively. \( A_r \) can also be written as

\[
n_z = n_0^2 \frac{\rho_r}{\rho_0} (n_1 - n_0)Y_rY_n^* \quad (22)
\]

Here, \( r \) and \( r_Y \) are the fractional modal powers in the corner regions for bent planar waveguides along \( x \) and \( y \) axes and they are given as (23) and (24) shown at the top of the next page where, \( x \) and \( y \) are the radiation points in the outer cladding for the two bent planar waveguides.

III. RESULTS AND DISCUSSION

Following the above analysis, we obtained the change in the effective indexes of the first two modes as a function of \( \theta \) for different RCW's with a fixed bending radius (\( \alpha_p = 1.5 \times 10^{-4} \)). Waveguides with both low as well as high core-cladding contrast and \( V \)-values have been considered. The values of various parameters considered are given as low-contrast: 722 = 1.457, \( n_1 = 1.004 \times 722 \), \( a/b = 2 \); high-contrast: \( n_2 = 1.470, m = 1.485, a/b = 2 \).

The values of \( n_1 \) and \( n_2 \) for low contrast \((121 - n_2^1)/n_1 = 0.4\%\) waveguide corresponds to the elliptic-core fiber used by Castro et al. [5] in their experiment on fiber accelerometer. The values of \( n_1 \) and \( n_2 \) for the high contrast waveguide are selected to be the same as in our earlier papers [8], [16] on dual-mode waveguides, giving the contrast \((n_1 - n_2)/n_1 \leq 1.0\%\).

Figs. 2 and 3 show the variation of bending-induced


\[ \int_{-\rho}^{\rho} (1 + x/\sec 9)^{-1}X^2(x)dx + \int_{-\mu}^{\mu} (1 + x/\sec 8)^{-1}X^2(x)dx \]

and

\[ \int_{-\rho}^{\rho} x/\sec 9 X^2(x)dx + \int_{-\mu}^{\mu} x/\sec 8 X^2(x)dx \]

(23)

\[ \int_{-\rho}^{\rho} (1 + y/\sec 6)^{-1}Y^2(y)dy + \int_{-\mu}^{\mu} (1 + y/\sec 9)^{-1}Y^2(y)dy \]

(24)

\[ \Gamma_{\nu} = \int_{-\rho}^{\rho} (1 + y/\sec 8)^{-1}Y^2(y)dy \]

Fig. 2. Bending-induced changes in the effective index of the first mode \((\Delta n_1)\) as a function of \(\theta\), for low contrast (continuous curve) and for high contrast (dashed curve) waveguides for \(V = 2.5, 3,\) and \(4\).

Changes \((\Delta n_1, \Delta n_2)\) in the effective indexes of the first and the second mode, respectively, for three different \(V = k_{anf} - nz)^{1/2}\) values, namely, \(2.5, 3,\) and \(4\). The continuous and the dashed curves correspond to the low and high contrast waveguides, respectively. These figures indicate that both \(\Delta n_1\) and \(\Delta n_2\) are sensitive functions of \(\theta\), which is due to the fact that the net change in \(n_1\) or \(n_2\) depends on the change in \(B_1\) and \(B_2\), both and which in turn depends on \(\theta\). For example, the change in \(B_1\) is zero for \(\theta = \pi/2\) and maximum for \(\theta = 0\), while the reverse is true for the change in \(B_2\). Another important observation is that both \(\Delta n_1\) and \(\Delta n_2\), have either a maxima or minima at \(\theta = 0\) and \(\pi/2\), i.e., the variation of \(\Delta n_1\) and \(\Delta n_2\), with \(\theta\) is symmetrical around \(\theta = 0\) (major) and \(\theta = \pi/2\) (minor) axis.

It should be noted that as far as the sensitivity of a dual-mode fiber optic sensor is concerned, it is the change in \(\Delta n_1 - \Delta n_2\) which is important rather than \(\Delta n_1, \Delta n_2\). Fig. 4 shows the variation of \(\Delta n_1 - \Delta n_2\) as a function of \(\theta\) for \(V = 2.5, 3,\) and \(4\) for the low contrast (continuous curve) and for high contrast (dashed curve) waveguides. These results show that in all the cases \((\Delta n_1 - \Delta n_2)\) is maximum for \(\theta = 0\), irrespective of the position of maxima in \(\Delta n_1\) or \(\Delta n_2\) versus \(\theta\). This means that the sensor using a rectangular/elliptic bent dual-mode waveguide will be most sensitive when bending due to physical parameters is induced in the plane of major axis.

The above observation is attributed to the fact that the second mode differs with the fundamental mode most in the direction of major axis \(0 - 0\), as the waveguide under consideration is dual-moded along the major axis and single-moded along the minor axis \((\theta = \pi/2)\). As a result, the difference between the bending-induced changes in \(n_1\) and \(n_2\), and hence \(\Delta n_1 - \Delta n_2\), is maximum for \(\theta = 0\). Fig. 4 also shows that as the \(V\)-number of the waveguide decreases, the value of \((\Delta n_1 - \Delta n_2)\) first decreases and then starts increasing in the opposite direction. In order to understand this effect, one has to keep in mind how the modal-power distribution in a bent waveguide is affected as the \(V\)-number changes. As discussed in our earlier paper [15], the modal power for the fundamental mode always shifts away from the
In order to obtain $SP_{\omega}, SP_{\lambda}$ and $X'$ we substitute $X, P_x, B_x$ from (12), (14), and (15) in (10). Equating the terms of different orders in $p^l$ one obtains:

$$d^2X_0 + [k_0^2 n'(x) - \beta_{20}^2] X_0 = 0$$

$$d^2X' + (k_0^2 n'(x) - \beta_{20}^2) X' = 2\left\{\beta_{31} \delta \beta_{21} - \frac{\delta^2\beta}{\sec \theta}\right\} X_0$$

$$d^2X'' + \left[k_0^2 n''(x) - \beta_{20}^2\right] X'' = 2\left\{\beta_{30} \delta \beta_{21} + \beta_{20} \delta \beta_{21}\right\} X_0 + \left\{\delta\beta_{21} - (\beta_{30} \delta \beta_{21} + \beta_{20} \delta \beta_{21})\right\} X_0$$

The solutions of (A1), which describes the modes of a straight planar waveguide characterized by $n'(x)$, are well known and are given by

$$X_0 = \begin{cases} \cos(u_x + 4) \exp\left[-\%x(x - a)\right] & x > a \\ \cos(u_x - \phi_x) \exp\left[\frac{\nu_x}{2}(x + a)\right] & x < -a \end{cases}$$

If we now multiply (A2) by $X_0$ and (A1) by $X'$ and subtract, we get

$$2\left\{\beta_{30} \delta \beta_{21} - \beta_{20}^2\right\} X_0 = \frac{d}{dx}\left\{X_0 \frac{dX'}{dx} - X' \frac{dX_0}{dx}\right\}$$

Integrating it over $x$ form $-\infty$ to 0 gives

$$\beta_{20} \sec \theta \int_{-\infty}^{0} X_0^2 \frac{dX}{dx} = 0.$$ 

Similarly

$$\delta \beta_{21} = 0.$$ 

Substituting $SP_{\omega}$, in (A2) gives

$$X'' = \left[A_1 \cos\left(\frac{u_x}{\alpha} x + \phi_x\right) + A_2 \sin(9x) + A_3 \left(\frac{u_x}{\alpha} x + 4\right) \sin(9x) + 4\right] x < -a$$

$$X' = \left[A_1 \cos\left(\frac{u_x}{\alpha} x + \phi_x\right) + A_2 \sin(9x) - u_x(9) \sin(9x) + 4\right] x < -a$$

$$X_0 = \left[A_1 \cos\left(\frac{u_x}{\alpha} x + \phi_x\right) + A_2 \sin(9x) - u_x(9) \sin(9x) + 4\right] x > a$$

$$X_0 = \left[A_1 \cos\left(\frac{u_x}{\alpha} x + \phi_x\right) + A_2 \sin(9x) - u_x(9) \sin(9x) + 4\right] x < -a.$$
of \( X' \) and \( dX'/dx \) at \( bx = a \). Solving for \( AI, A_0, \) As, and \( A_4 \) and putting them back in the above equations, we get the first order field correction \( \left( X'_0 \right) \) in the form as given in (18).

Also multiplying (A3) by \( X' \), and (AI) by \( X'' \) and subtracting, we get

\[
2 \left[ \beta_0 \delta x_2 X_0'^2 - \frac{\beta_0^2}{\sec \theta} \frac{x}{X_0} \right] = \frac{d}{dx} \left[ X_0 \frac{dX''}{dx} - X'' \frac{dX_0}{dx} \right].
\]

Integrating it over \( x \) from \(-\infty\) to \( 0 \) gives

\[
\delta \beta_{22} = \frac{P_1 \int_{-\infty}^{0} xX_0'X'dx}{\int_{-\infty}^{0} X'x'dx}.
\]

Using the expressions of \( X_0 \) and \( X' \) and integrating (A5) we get \( \delta X_2 \) as given in (16).

Similarly, the expressions for \( Y'' \) and \( SP_{22} \) are also obtained.

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REFERENCES


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